JOURNAL OF APPROXIMATION THEORY **45**, 358–362 (1985)

# The Main Roots of the Euler–Frobenius Polynomials

M. REIMER

*Lehrstuhl Mathematik III, Universität Dortmund, Postfach 500500, D-4600 Dortmund 50, West Germany*

*Communicated by G. Meinardus*

Received July 18, 1984

We investigate the behavior of the largest root $\leqslant -1$ of an Euler–Frobenius polynomial. This root determines the convergence/divergence of a cardinal Lagrange spline series. Asymptotic representations are obtained in the most important cases. © 1985 Academic Press, Inc.

Recently [4], we investigated the convergence of interpolating cardinal Lagrange spline series

$$\sum_{j \in \mathbb{Z}} y_j l_j(x), \qquad x \in \mathbb{R},\ y_j \in \mathbb{R}, \qquad \text{for} \quad j \in \mathbb{Z},$$

where $l_j$ denotes the Lagrange spline of degree $m \in \mathbb{N}$ with respect to the integer grid and node $j$. If $m$ is odd, $m = 2r+1$, then we have convergence/divergence if

$$\overline{\lim_{|j| \to \infty}} \sqrt[|j|]{|y_j|} \underset{(>)}{<} |z_r^{(m)}(0)| \tag{1}$$

where $z_r^{(m)}(0) < -1$ denotes the main root of the Euler (-Frobenius) polynomial

$$H_m(t, z) = (1-z)^{m+1} \left( t + z \frac{\partial}{\partial z} \right)^m \frac{1}{1-z} \tag{2}$$

for the parameter value $t = 0$.

It is well known [2] that the roots $z_j^{(m)}(t)$ of $H_m(t, z)$, $t \in [0, 1]$, are simple and located on the negative axis, say,

$$-\infty \leqslant z_1^{(m)}(t) < z_2^{(m)}(t) < \cdots < z_m^{(m)} \leqslant 0$$



0021-9045/85 $3.00

where

$$z_r^{(2r+1)}(0) < -1 < z_{r+1}^{(2r+1)}(0).$$

$$z_r^{(2r)}(\tfrac{1}{2}) < -1 < z_{r+1}^{(2r)}(\tfrac{1}{2}).$$

Generally we call the greatest root of $H_m(t, z)$ which does not exceed $-1$ the main root of $H_m(t, z)$, and we denote it by $\zeta_m(t)$ (for $t \in [0, 1]$, $r \in \mathbb{N}$).

Hence we have

$$\zeta_m(0) = z_r^m(0) \qquad \text{if} \quad m = 2r+1, r \in \mathbb{N},$$

$$\zeta_m(\tfrac{1}{2}) = z_r^m(\tfrac{1}{2}) \qquad \text{if} \quad m = 2r, r \in \mathbb{N}.$$

If we follow the lines of the proof for (1) in [4] then we obtain in case of a shifted interpolation grid $\lambda + \mathbb{Z}$, $\lambda \in [0, \frac{1}{2}]$, a condition similar to (1), where the right-hand side is to be replaced by

$$\min\{|\zeta_m(\lambda)|, |\zeta_m(1-\lambda)|\} \tag{3}$$

(cf. also [5]).

As indicated in [3], each root $z_j^m(t)$ is a monotonously decreasing function for $0 \leqslant t \leqslant 1$. This, together with $H_m(t, z) = z^m H_m(1-t, z^{-1})$, cf. [2], implies

$$\min\{|\zeta_m(\lambda)|, |\zeta_m(1-\lambda)|\} \leqslant |\zeta_m(0)| \text{ for } m = 2r+1,$$
$$\leqslant |\zeta_m(\tfrac{1}{2})| \text{ for } m = 2r.$$

Hence the greatest possible radius of convergence is obtained for $\lambda = 0$ if $m$ is odd and for $\lambda = \frac{1}{2}$ if $m$ is even. For this reason we study the asymptotic behavior of $\zeta_{2r+1}(0)$ and $\zeta_{2r}(\frac{1}{2})$, respectively.

In case of $m = 2r+1$ we get ready information by a result of Sobolev [6]. If we define the Euler polynomials $E_{m-1}$ by

$$zE_{m-1}(z) = (1-z)^{m+1} \left(z \frac{d}{dz}\right)^m \frac{z}{(1-z)^2}$$

for $m \in \mathbb{N}$, then we have obviously

$$zE_{m-1}(z) = H_m(0, z).$$

Hence [6, Theorem 2] yields

$$\zeta_{2r+1}(0) = -\exp \pi \left[ \tan \frac{\pi}{4(r+1)} + O(\eta^r) \right] \tag{4}$$

as $r \to \infty$, where the constant $0 < \eta < 1$ has to be chosen such that

$$\sum_{n=2}^{\infty} \frac{1}{(2n-1)^{2r+2}} = O(\eta^r). \tag{5}$$

We are going to treat the case where $\lambda = \frac{1}{2}$ and $m = 2r$, $r \in \mathbb{N}$. By a representation formula due to ter Morsche [2] we obtain (generally)

$$\begin{aligned} H_m(\lambda, -e^x) &= (1+e^x)^{m+1} \left(\lambda + \frac{d}{dx}\right)^m \frac{1}{1+e^x} \\ &= (1+e^x)^{m+1} e^{-\lambda x} \Phi_m(\lambda, x) \end{aligned} \tag{6}$$

where

$$\Phi_m(\lambda, x) = \left(\frac{d}{dx}\right)^m \frac{e^{\lambda x}}{1+e^x}. \tag{7}$$

By (6), $H_m(\frac{1}{2}, -e^x)$ vanishes if and only if

$$\Phi_m(\tfrac{1}{2}, x) = \left(\frac{d}{dx}\right)^m \frac{1}{\cosh(x/2)} \tag{8}$$

vanishes. Now, let $x = \pi t$. Then

$$\Phi_m(\tfrac{1}{2}, \pi t) = \left(\frac{1}{\pi}\right)^{m+1} \left(\frac{d}{dt}\right)^m \frac{\pi}{\cosh(\pi t/2)}.$$

Let us modify Sobolev's ideas as follows:

By the use of the partial fraction expansion of $\pi/\cos(\pi x/2)$, cf. [1, p. 232], e.g., we obtain

$$\pi^{m+1}\Phi_m(\tfrac{1}{2}, \pi t) = \left(\frac{d}{dt}\right)^m \left\{\pi + \sum_{n=-\infty}^{+\infty} (-1)^n \left[\frac{2}{it-2n+1} + \frac{2}{2n-1}\right]\right\}$$

or

$$\frac{1}{2}\pi^{m+1}\Phi_m\left(\frac{1}{2}, \pi t\right) = m!\, i^m \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(it-2n+1)^{m+1}},$$

where it suffices again to make use of the members for $n = 0, 1$ in the sum. The result is that

$$z_j^{(2r)}\left(\frac{1}{2}\right) = -\exp \pi \left[\tan \frac{2(j-r)+1}{2r+1} \frac{\pi}{2} + O(\eta^r)\right] \tag{9}$$

holds for $j=1,\ldots,2r$ as $r\to\infty$ with the same constant $\eta$ as in (5). Equations (4) and (9) together yield the

THEOREM. *For the main roots of the Euler–Frobenius polynomials* $H_{2r+1}(0,z)$ *and* $H_{2r}(\frac{1}{2},z)$, *respectively, the asymptotic relations*

$$\zeta_{2r}\left(\frac{1}{2}\right)=-\left\{1+\frac{\pi^2}{4r+4}+O\left(\frac{1}{r^2}\right)\right\},$$

$$\zeta_{2r+1}(0)=-\left\{1+\frac{\pi^2}{4r+2}+O\left(\frac{1}{r^2}\right)\right\}$$

*hold as r tends to* $\infty$.

*Final Remark.* From Sobolev [6, Theorem 1] it follows, in addition, that

$$\zeta_3(0)<\zeta_5(0)<\cdots<-1$$

holds. Hence, for odd $m>1$, the cubic splines provide for the greatest possible radius of convergence.

Note that

$$\zeta_2\left(\frac{1}{2}\right)=-3-\sqrt{8}=-5.82847\ldots,$$

$$\zeta_3(0)=-2-\sqrt{3}=-3.73205\ldots,$$

$$\zeta_4\left(\frac{1}{2}\right)=-\frac{6+\sqrt{3(7-\sqrt{19})}}{6-\sqrt{3(7-\sqrt{19})}}=-2.76746\ldots,$$

$$\zeta_5(0)=\frac{2+\sqrt{15+\sqrt{105}}}{2-\sqrt{15+\sqrt{105}}}=-2.32247\ldots.$$

Generally we can interpolate data $y_j$, $j\in\mathbb{Z}$, by a cardinal Lagrange spline series of arbitrary high degree $m$ only if they are growing for $j\to\pm\infty$ slower than by any exponential rate.


## ACKNOWLEDGMENTS

The author is indebted to the referee for a valuable hint.


## REFERENCES


1. H. BEHNKE AND F. SOMMER, "Theorie der analytischen Funktionen einer komplexen Veränderlichen," Springer-Verlag, Berlin/Göttingen/Heidelberg, 1955.

2. H. TER MORSCHE, On the existence and convergence of interpolating periodic spline functions, *in* "Spline-Funktionen" (K. Böhmer, G. Meinardus, and W. Schempp, Eds.), pp. 177–195, Bibliographisches Institut, Mannheim, 1974.
3. M. REIMER, Extremal spline bases, *J. Approx. Theory* **36** (1982), 91–98.
4. M. REIMER, The radius of convergence of a cardinal Lagrange spline series of odd degree, *J. Approx. Theory* **39** (1983), 289–294.
5. D. SIEPMANN, Kardinale Spline-Interpolation bezüglich äquidistant verteilter Knoten, Dissertation, Dortmund, 1984.
6. S. L. SOBOLEV, On the roots of Euler polynomials, *Dokl. Akad. Nauk SSSR* **235** (1977). [*Soviet Math. Dokl.* **18** (1977), 935–938].